Non-standard quantum $so(2, 2)$ and beyond

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Abstract. A new 'non-standard' quantization of the universal enveloping algebra of the split (natural) real form $so(2, 2)$ of D_2 is presented. Some (classical) graded contractions of $so(2, 2)$ associated to a $\mathbb{Z}_2 \cdot \times \mathbb{Z}_2$ grading are studied, and the automorphisms defining this grading are generalized to the quantum case, thus providing quantum contractions of this algebra. This produces a new family of 'non-standard' quantum algebras. Some of these algebras can be realized as $(2 + 1)$ kinematical algebras; we explicitly introduce a new deformation of Poincaré algebra, which is naturally linked to the null plane basis. Another realization of these quantum algebras as deformations of the conformal algebras for the two-dimensional Euclidean, Galilei and Minkowski spaces is given, and its new properties are emphasized.

1. Introduction

Quantum algebras can be understood as deformations of Lie bi-algebras. For a semisimple Lie algebra g, all its Lie bi-algebras are coboundary structures coming from classical r-matrices. Thus, the classification of r-matrices provides a first description of the inequivalent quantizations of g. In general, r-matrices can be either *non-degenerate* (i.e. a skew solution of the modified Yang–Baxter equation (MYBE)) or *degenerate* (i.e. a skew solution of the classical Yang–Baxter equation (CYBE)) [1]. All non-degenerate r-matrices for simple Lie algebras were obtained by Belavin and Drinfel'd [2].

For $sl(2, \mathbb{R})$ there exist three classes of Lie bi-algebras [1]: one of them ($r = 2J_+ \wedge J_-$) is non-degenerate and corresponds to the well known Drinfel'd–Jimbo quantum deformation, hereafter called *standard* deformation. The two remaining solutions are generated by the trivial one $r = 0$ and by $r = J_3 \wedge J_+$, respectively. The quantization of the latter has been recently worked out by Ohn [3], and provides a new kind of *non-standard* quantum deformation of $sl(2, \mathbb{R})$.

Many efforts have been also devoted to obtaining quantum non-semisimple algebras from contraction of the standard deformation, but—to our knowledge—no contractions of this non-standard quantum algebra $U_z sl(2, \mathbb{R})$ or related algebras have been explored so far. The aim of this paper is twofold: firstly, to construct such a non-standard quantum $so(2, 2)$ by using the prescription $U_z so(2, 2) \simeq U_z so(2, 1) \oplus U_{-z} so(2, 1)$ [4, 5] as applied to the Ohn non-standard quantization [3] for the two $so(2, 1) \simeq sl(2, \mathbb{R})$ copies, and to compare it with the standard quantum $so(2, 2)$. Secondly, to study the quantum contractions of the standard and non-standard quantum $so(2, 2)$ algebras. In particular, new 'non-standard' quantum kinematical (de Sitter, anti-de Sitter and Poincaré) algebras in $(2 + 1)$ dimensions are obtained, as well as new quantum conformal algebras in two dimensions.

The usual way to deal with contractions is the Inönü–Wigner (IW) [6] scheme, where a single parameter is made to go towards some singular limit. Recently, a more comprehensive

contraction method based on the preservation of a given grading of the Lie algebra has been developed by Moody and Patera [7]; this method includes both the IW contractions as well as some kind of general 'Weyl unitary trick', which relates different real forms of the same complex algebra. In this paper we adopt this graded contraction point of view, and we consistently consider only real forms of Lie algebras. Therefore, we start in section 2 with a (real) grading of the natural non-compact real form $so(2,2)$ determined by a set of commuting involutive automorphisms. A family of graded contracted algebras depending on three real parameters, $g_{(\mu_1,\mu_2,\mu_3)}$, is naturally distinguished, and includes the algebra $so(2,2)$ as $g_{(+1,+1,+1)}$. The information conveyed by these automorphisms is relevant in order to classify the quantum deformations of the graded contractions of $so(2,2)$.

In principle, we could try to quantize these graded contractions by using the two different (*standard* and *non-standard*) $so(2,2)$ deformations which are presented in section 3. Furthermore, as is shown in section 4, it turns out that in each case not all $U_z so(2,2)$ contractions are allowed, but only some well defined family. Some of the results obtained contracting the standard deformation to the family $g_{(\mu_1,+1,\mu_3)}$ can be related to those already known (see e.g. [8]), but we get a new set of non-standard deformations for the family of algebras $g_{(\mu_1,\mu_2,+1)}$ which include new deformations of the most interesting $(2+1)$ kinematical algebras: de Sitter and Poincaré algebras.

When these algebras are understood as conformal algebras in two dimensions, we get new quantum non-standard conformal algebras, which, like the standard ones, contains Hopf subalgebras generated by isometries of the space augmented just with dilations. However, these quantum algebras have completely new features as compared with the known ones (see e.g. [9]). In particular, both translations appear in a completely symmetrical way, and both of them are *primitive*. All these results are presented in section 5.

2. Graded contractions of $so(2, 2)$

Let us recall briefly the theory of graded contractions of Lie algebras [7]. Suppose L is a *real* Lie algebra, graded by an abelian finite group Γ whose product is denoted additively. The grading is a decomposition of the vector space structure of L as

$$L = \bigoplus_{\mu \in \Gamma} L_\mu \tag{2.1}$$

such that for $x \in L_\mu$ and $y \in L_\nu$ then $[x, y]$ belongs to $L_{\mu+\nu}$. This is written as

$$[L_\mu, L_\nu] \subseteq L_{\mu+\nu} \qquad \mu, \ \nu, \ \mu + \nu \in \Gamma. \tag{2.2}$$

A *graded contraction* of the Lie algebra L is a Lie algebra $L^{(\varepsilon)}$ with the same vector space structure as L, but Lie brackets for $x \in L_\mu$, $y \in L_\nu$ modified as

$$[x, y]_\varepsilon := \varepsilon_{\mu,\nu} [x, y] \tag{2.3}$$

where the *contraction parameters* $\varepsilon_{\mu,\nu}$ are real numbers such that $L^{(\varepsilon)}$ is indeed a Lie algebra; this implies that they should satisfy the *contraction equations*

$$\varepsilon_{\mu,\nu} = \varepsilon_{\nu,\mu}$$
$$\varepsilon_{\mu,\nu} \, \varepsilon_{\mu+\nu,\sigma} = \varepsilon_{\mu,\nu+\sigma} \, \varepsilon_{\nu,\sigma} \tag{2.4}$$

for all *relevant* values of indices. Each set of parameters ε which is a solution of (2.5) defines a contraction; two contractions $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are equivalent if they are related by

$$\varepsilon^{(2)}_{\mu,\nu} = \varepsilon^{(1)}_{\mu,\nu} \frac{r_\mu r_\nu}{r_{\mu+\nu}} \tag{2.5}$$

(without summation over repeated indices) where the r's are *non-zero* real numbers which should be thought of as *scaling factors* of the grading subspaces.

Even if the contraction parameters associated to *any* pair of elements μ and ν in Γ seem to appear in (2.5), many of them will not, because it could happen that in the non-contracted algebra, all the elements $x \in L_\mu$ commute with the elements $y \in L_\nu$; the parameters $\varepsilon_{\mu,\nu}$ corresponding to $[L_\mu, L_\nu] = 0$ are irrelevant and (2.5) which contain such parameters do not appear.

Let us consider two copies of $so(2, 1)$, each one with basis $\{J_3^l, J_\pm^l\}$ $(l = 1, 2)$, and commutation relations given by

$$[J_3^l, J_\pm^l] = \pm 2 J_\pm^l \qquad [J_+^l, J_-^l] = J_3^l. \tag{2.6}$$

The set of generators $\{J_3, J_\pm, N_3, N_\pm\}$ defined by

$$J_m = J_m^1 + J_m^2 \qquad N_m = J_m^1 - J_m^2 \qquad m = +, -, 3 \tag{2.7}$$

closes a $so(2, 2)$ Lie algebra with commutation rules

$$
\begin{aligned}
&[J_3, J_\pm] = [N_3, N_\pm] = \pm 2 J_\pm && [J_3, N_\pm] = [N_3, J_\pm] = \pm 2 N_\pm \\
&[J_+, J_-] = [N_+, N_-] = J_3 && [J_\pm, N_\mp] = \pm N_3 \\
&[J_m, N_m] = 0 && m = +, -, 3.
\end{aligned}
\tag{2.8}
$$

The two second-order Casimirs for this algebra are

$$C_1 = \tfrac{1}{2} J_3^2 + \tfrac{1}{2} N_3^2 + J_+ J_- + J_- J_+ + N_+ N_- + N_- N_+ \tag{2.9}$$

$$C_2 = \tfrac{1}{2} J_3 N_3 + J_+ N_- + J_- N_+. \tag{2.10}$$

The Lie algebra mappings defined by

$$S^{(\epsilon_1, \epsilon_2)} : (J_3, J_\pm, N_\pm, N_3) \to (J_3, \epsilon_2 J_\pm, \epsilon_1 \epsilon_2 N_\pm, \epsilon_1 N_3) \tag{2.11}$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$, are four commuting involutive automorphisms of $so(2, 2)$ which generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading of the $so(2, 2)$ Lie algebra. In particular, $S^{(+,+)} = 1$, and sometimes we will denote $S^{(-,+)} \equiv S_1$, $S^{(+,-)} \equiv S_2$ and $S^{(-,-)} \equiv S_3$. With the usual notation for the grading subspaces according as their elements are either invariant or anti-invariant under S_1 and S_2, we have

$$L_{00} = \langle J_3 \rangle \qquad L_{01} = \langle J_+, J_- \rangle \qquad L_{10} = \langle N_3 \rangle \qquad L_{11} = \langle N_+, N_- \rangle. \tag{2.12}$$

The contraction coefficients $\varepsilon_{00,00}$, $\varepsilon_{00,10} = \varepsilon_{10,00}$ and $\varepsilon_{10,10}$ are irrelevant, as the associated pairs of grading subspaces already commute. The remaining coefficients

$$
\begin{aligned}
\gamma &= \varepsilon_{00,01} & \chi &= \varepsilon_{00,11} \\
\alpha &= \varepsilon_{01,01} & \beta &= \varepsilon_{11,11} \\
\delta &= \varepsilon_{01,11} & \xi &= \varepsilon_{01,10} & \tau &= \varepsilon_{11,10}
\end{aligned}
$$

should satisfy the contraction equations (2.5):

$$
\begin{aligned}
(\gamma - \chi)\xi = (\gamma - \chi)\tau &= (\gamma - \chi)\delta = 0 \\
\alpha\chi = \xi\delta \qquad \delta\tau &= \gamma\beta \qquad \xi\beta = \alpha\tau.
\end{aligned}
\tag{2.13}
$$

A naturally distinguished set of solutions of these equations is obtained by requiring $\gamma = \chi \neq 0$; each such a solution is equivalent to a solution with $\gamma = \chi = 1$; the *general* solution of this special case can be expressed in terms of three real constants, μ_1, μ_2 and μ_3, by means of

$$\alpha = \mu_2 \mu_3 \qquad \beta = \mu_1 \mu_2 \qquad \tau = \mu_1 \qquad \xi = \mu_3 \qquad \delta = \mu_2. \tag{2.14}$$
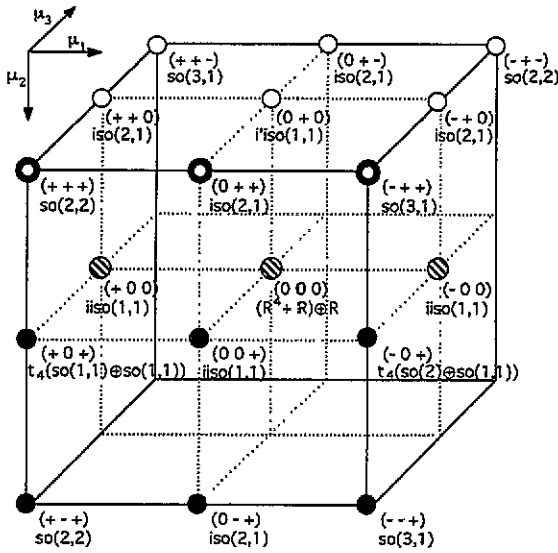
Figure 1. Algebras in the family $g_{(\mu_1,\mu_2,\mu_3)}$. Each point corresponds to a set of values (μ_1, μ_2, μ_3) of the contraction constants assumed to be normalized to ± 1 and 0. Open circles correspond to algebras in the family (a); full circles correspond to algebras in the family (b). The three algebras in the upper front edge belong to both families. Shaded circles are in families (a0) or (b0) but not in (a) or (b). All remaining positions in the diagram are equivalent as graded contractions to some algebra already depicted.

The contracted algebras will be denoted by $g_{(\mu_1,\mu_2,\mu_3)}$. Different choices of the constants may correspond to equivalent graded contractions (and therefore to isomorphic algebras). It is a simple exercise to check that, first, the graded contraction $g_{(\mu_1,\mu_2,\mu_3)}$ is equivalent to $g_{(-\mu_1,-\mu_2,-\mu_3)}$; the scale factors carrying out this equivalence correspond to the reversal of N_3, and second, any solution $g_{(\mu_1,\mu_2,\mu_3)}$ is equivalent to one where each μ_1, μ_2, μ_3 can take on the values $\{+1, 0, -1\}$.

Therefore, the equivalence classes of graded contractions with $\gamma = \chi \neq 0$ can be represented as the vertices, middle points of edges, middle points of faces, and centre of a cube with 'antipodal' identification. This means a total of 14 classes of non-equivalent graded contractions, which are depicted in figure 1, and explicitly listed in table 1 below.

The most relevant algebras in this list are $so(2, 2)$, $iso(2, 1)$ and $so(3, 1)$; each of them appears several times, and can be either interpreted as the algebras of isometries of the $(2 + 1)$ anti-de Sitter, Minkowski and de Sitter spaces, or, alternatively, as conformal algebras of $(1 + 1)$ Minkowski and Galilean planes, and of the 2D Euclidean plane. In order to highlight this last interpretation, and at the same time to distinguish each of the graded contraction algebras $g_{(\mu_1,\mu_2,\mu_3)}$ from the $so(2, 2)$ we started with, we choose a new naming J, P_1, P_2, C_1, C_2 and D for the generators of $g_{(\mu_1,\mu_2,\mu_3)}$, in such a way that for $g_{(1,1,1)}$ they are related to the ones of $so(2, 2)$ by

$$J = \tfrac{1}{2}N_3 \qquad P_1 = J_+ \qquad P_2 = N_+ \qquad C_1 = -J_- \qquad C_2 = N_- \qquad D = \tfrac{1}{2}J_3 .$$

(2.15)

The commutation relations and Casimirs of $g_{(\mu_1,\mu_2,\mu_3)}$ in this basis are

$$
\begin{aligned}
&[J, P_1] = \mu_3 P_2 \qquad [J, P_2] = \mu_1 P_1 \qquad [P_1, P_2] = 0 \qquad [D, P_i] = P_i \\
&[J, C_1] = \mu_3 C_2 \qquad [J, C_2] = \mu_1 C_1 \qquad [C_1, C_2] = 0 \qquad [D, C_i] = -C_i \\
&[P_1, C_1] = -2\mu_2\mu_3 D \qquad [P_1, C_2] = 2\mu_2 J \qquad [D, J] = 0 \\
&[P_2, C_1] = -2\mu_2 J \qquad [P_2, C_2] = 2\mu_1\mu_2 D
\end{aligned}
$$

(2.16)

$$\mathcal{C}_1 = \mu_2 J^2 + \mu_1\mu_2\mu_3 D^2 - \tfrac{1}{2}\mu_1(P_1 C_1 + C_1 P_1) + \tfrac{1}{2}\mu_3(P_2 C_2 + C_2 P_2)$$

(2.17)

$$\mathcal{C}_2 = \mu_2 J D + \tfrac{1}{2}(P_1 C_2 - C_1 P_2) .$$

(2.18)

It is therefore clear that $g_{(1,1,1)}$ and $g_{(-1,1,1)}$ are the conformal algebras of $(1 + 1)$ Minkowski space and the algebra of the group of Möbius transformations in the Euclidean plane; the names of the generators have been chosen to underline this fact.

The involutive automorphisms S_i act on the generators of $g_{(\mu_1, \mu_2, \mu_3)}$ as

$$S_1(J, P_1, P_2, C_1, C_2, D) = (-J, P_1, -P_2, C_1, -C_2, D)$$
$$S_2(J, P_1, P_2, C_1, C_2, D) = (J, -P_1, -P_2, -C_1, -C_2, D) \quad (2.19)$$
$$S_3(J, P_1, P_2, C_1, C_2, D) = (-J, -P_1, P_2, -C_1, C_2, D)$$

and have associated IW contractions, ι_i, as the limit $\lambda_i \to 0$ of the Lie algebra automorphisms

$$\iota_1(J, P_1, P_2, C_1, C_2, D) := (\lambda_1 J, P_1, \lambda_1 P_2, C_1, \lambda_1 C_2, D)$$
$$\iota_2(J, P_1, P_2, C_1, C_2, D) := (J, \lambda_2 P_1, \lambda_2 P_2, \lambda_2 C_1, \lambda_2 C_2, D) \quad (2.20)$$
$$\iota_3(J, P_1, P_2, C_1, C_2, D) := (\lambda_3 J, \lambda_3 P_1, P_2, \lambda_3 C_1, C_2, D)$$

which appear as the graded contractions $(\mu_1, \mu_2, \mu_3) = (0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$.

The interchange $(P_1, C_1) \leftrightarrow (P_2, C_2)$ is a Lie algebra automorphism $g_{(\mu_1, \mu_2, \mu_3)} \to g_{(\mu_3, \mu_2, \mu_1)}$, so the list of 14 non-equivalent graded contractions reduces, up to isomorphisms, to eight Lie algebras. Instead of working with this list, two different choices of representatives of the equivalence classes of graded contractions (named (a) and (b)) will be useful in this paper:

Type (a): $g_{(\mu_1, +1, \mu_3)}$ where $\mu_1, \mu_3 \in \{+1, 0, -1\}$

Type (a0): $g_{(\mu_1, 0, \mu_3)}$ where $(\mu_1, \mu_3) \in \{(1, 1), (0, 1), (-1, 1), (1, 0), (0, 0)\}$ (2.21)

Type (b): $g_{(\mu_1, \mu_2, +1)}$ where $\mu_1, \mu_2 \in \{+1, 0, -1\}$

Type (b0): $g_{(\mu_1, \mu_2, 0)}$ where $(\mu_1, \mu_2) \in \{(1, 1), (0, 1), (-1, 1), (1, 0), (0, 0)\}$.   (2.22)

We shall keep in mind that each of the families a/a0 or b/b0 contain *all* graded contractions up to equivalence. In figure 1, type (a) and type (b) algebras appear on the top and front faces of the cube, respectively. Note that the three algebras $g_{(\mu_1, +1, +1)} (\simeq so(2, 2), iso(2, 1), so(3, 1))$ on the upper front edge are common to both sets of representatives.

A very concise and practical way to describe this family of graded contractions of $so(2, 2)$ is to get the algebra $g_{(\mu_1, \mu_2, \mu_3)}$ by means of the formal transformation (compare with Man'ko and Gromov [10]) applied to the $so(2, 2) \simeq g_{(+1, +1, +1)}$ generators:

$$(J, P_1, P_2, C_1, C_2, D) = \Gamma^{(\mu_1, \mu_2, \mu_3)}(N_3/2, J_+, N_+, -J_-, N_-, J_3/2)$$
$$:= (\sqrt{\mu_1 \mu_3} N_3/2, \sqrt{\mu_2 \mu_3} J_+, \sqrt{\mu_1 \mu_2} N_+, -\sqrt{\mu_2 \mu_3} J_-, \sqrt{\mu_1 \mu_2} N_-, J_3/2)$$
$$(2.23)$$

which is well defined as long as all μ_i are different from zero, and where the new generators close the algebra (2.16). This device has been extensively used by Gromov in a slightly different form which also uses dual numbers [11]. For our purposes it will suffice to use this formal replacement when all μ_i are different from zero, and to understand (2.23) when some μ_i goes to zero as the corresponding limit (as in (2.20)). It should be noted that each IW contraction parameter λ_i in (2.20) corresponds to a factor $\sqrt{\mu_i}$ in (2.23).

3. Quantum deformations of $Uso(2, 2)$

3.1. Two deformations of $Uso(2, 1)$

The two quantizations of the non-trivial Lie bi-algebras of $so(2, 1)$ are given by the following statements:

Proposition 1 (The standard quantization of so(2, 1)) [12]. The coproduct (Δ), counit (ϵ), antipode (γ) defined by

$$\Delta J_3 = 1 \otimes J_3 + J_3 \otimes 1 \qquad \Delta J_{\pm} = e^{-zJ_3} \otimes J_{\pm} + J_{\pm} \otimes e^{zJ_3} \qquad (3.1)$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{zJ_3} X e^{-zJ_3} \qquad \text{for} \quad X \in \{J_3, J_{\pm}\} \qquad (3.2)$$

together with the commutation rules

$$[J_3, J_{\pm}] = \pm 2J_{\pm} \qquad [J_+, J_-] = \frac{\sinh(2zJ_3)}{2z} \qquad (3.3)$$

quantize the $so(2, 1)$ Lie bi-algebra generated by the classical r-matrix $r = 2J_+ \wedge J_-$ and define the (standard) Hopf algebra $U_z^{(s)}so(2, 1)$.

Note r verifies the MYBE. The centre of $U_z^{(s)}so(2, 1)$ is generated by

$$C_z = \tfrac{1}{2} \cosh 2z \left(\frac{\sinh(zJ_3)}{z} \right)^2 + \frac{\sinh 2z}{2z}(J_+ J_- + J_- J_+) . \qquad (3.4)$$

Proposition 2 (The non-standard quantization of so(2, 1)) [3]. The coproduct, counit, antipode

$$\Delta J_+ = 1 \otimes J_+ + J_+ \otimes 1 \qquad \Delta J_m = e^{-zJ_+} \otimes J_m + J_m \otimes e^{zJ_+} \qquad m = -, 3 \qquad (3.5)$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{zJ_+} X e^{-zJ_+} \qquad \text{for} \quad X \in \{J_3, J_{\pm}\} \qquad (3.6)$$

and the commutation relations

$$[J_3, J_+] = 2\frac{\sinh(zJ_+)}{z} \qquad [J_+, J_-] = J_3 \qquad (3.7)$$

$$[J_3, J_-] = -J_- \cosh(zJ_+) - \cosh(zJ_+)J_- \qquad (3.8)$$

define the Hopf algebra $U_z^{(n)}so(2, 1)$ that quantizes the non-standard Lie bi-algebra structure generated by $r = J_3 \wedge J_+$.

In this case, the r-matrix satisfies the CYBE. Now, the centre of $U_z^{(n)}so(2, 1)$ is generated by

$$C_z = \tfrac{1}{2}J_3^2 + \frac{\sinh(zJ_+)}{z}J_- + J_-\frac{\sinh(zJ_+)}{z} + \tfrac{1}{2}\cosh^2(zJ_+) . \qquad (3.9)$$

It is easy to check that in both cases the r-matrix gives the first-order terms of the deformation: the antisymmetric part of the first order of the coproduct of a given generator is just the cocommutator defined by the corresponding r-matrix:

$$(\Delta - \sigma \circ \Delta)(X) = \delta(X) = [1 \otimes X + X \otimes 1, r] . \qquad (3.10)$$

3.2. Two deformations of Uso(2,2)

By using the invariance of $U_z so(2, 1)$ under the transformation $z \to -z$, we can write $U_z^{(m)}so(2, 2) = U_z^{(m)}so(2, 1) \oplus U_{-z}^{(m)}so(2, 1)$, where $m = n$ or $m = s$ according either to the non-standard or standard $so(2, 1)$ deformations. Therefore two different q-deformations of $so(2, 2)$ can be obtained in this way. The proofs of propositions 3–8 boils down to straightforward checking and will not be given.

Proposition 3 *(The standard quantization of so(2,2))*. The coproduct, counit, antipode:

$$\Delta J_3 = 1 \otimes J_3 + J_3 \otimes 1 \qquad \Delta N_3 = 1 \otimes N_3 + N_3 \otimes 1$$

$$\Delta J_\pm = e^{-zN_3/2} \cosh(zJ_3/2) \otimes J_\pm + J_\pm \otimes \cosh(zJ_3/2) e^{zN_3/2}$$
$$-e^{-zN_3/2} \sinh(zJ_3/2) \otimes N_\pm + N_\pm \otimes \sinh(zJ_3/2) e^{zN_3/2} \tag{3.11}$$
$$\Delta N_\pm = e^{-zN_3/2} \cosh(zJ_3/2) \otimes N_\pm + N_\pm \otimes \cosh(zJ_3/2) e^{zN_3/2}$$
$$-e^{-zN_3/2} \sinh(zJ_3/2) \otimes J_\pm + J_\pm \otimes \sinh(zJ_3/2) e^{zN_3/2}$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{zN_3} X e^{-zN_3} \qquad \text{for} \quad X \in \{J_3, J_\pm, N_3, N_\pm\} \tag{3.12}$$

and the commutation relations

$$[J_3, J_\pm] = [N_3, N_\pm] = \pm 2 J_\pm$$
$$[J_3, N_\pm] = [N_3, J_\pm] = \pm 2 N_\pm$$

$$[J_+, J_-] = [N_+, N_-] = \frac{1}{z} \sinh(zJ_3) \cosh(zN_3) \tag{3.13}$$

$$[J_\pm, N_\mp] = \pm\frac{1}{z} \sinh(zN_3) \cosh(zJ_3) \qquad [J_m, N_m] = 0 \qquad m = +, -, 3$$

define the Hopf algebra $U_z^{(s)} so(2, 2) = U_z^{(s)} so(2, 1) \oplus U_{-z}^{(s)} so(2, 1)$.

The classical r-matrix corresponding to this q-deformation is obtained as the difference of the r-matrices generating the two $U_z^{(s)} so(2, 1)$ components:

$$r = r_1^{(s)} - r_2^{(s)} = 2J_+^1 \wedge J_-^1 - 2J_+^2 \wedge J_-^2 = J_+ \wedge N_- + N_+ \wedge J_-. \tag{3.14}$$

This r-matrix verifies the MYBE and generates the first-order term in z of the coproduct (3.11).

Proposition 4 *(The non-standard quantization of so(2,2))*. The coproduct, counit, antipode

$$\Delta J_+ = 1 \otimes J_+ + J_+ \otimes 1 \qquad \Delta N_+ = 1 \otimes N_+ + N_+ \otimes 1$$
$$\Delta J_m = e^{-zN_+/2} \cosh(zJ_+/2) \otimes J_m + J_m \otimes \cosh(zJ_+/2) e^{zN_+/2}$$
$$-e^{-zN_+/2} \sinh(zJ_+/2) \otimes N_m + N_m \otimes \sinh(zJ_+/2) e^{zN_+/2} \tag{3.15}$$
$$\Delta N_m = e^{-zN_+/2} \cosh(zJ_+/2) \otimes N_m + N_m \otimes \cosh(zJ_+/2) e^{zN_+/2}$$
$$-e^{-zN_+/2} \sinh(zJ_+/2) \otimes J_m + J_m \otimes \sinh(zJ_+/2) e^{zN_+/2} \qquad m = 3, -$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{zN_+} X e^{-zN_+} \qquad \text{for} \quad X \in \{J_3, J_\pm, N_3, N_\pm\} \tag{3.16}$$

and the commutation relations

$$[J_3, J_+] = \frac{4}{z} \sinh(zJ_+/2) \cosh(zN_+/2)$$
$$[J_3, J_-] = -\{J_-, \cosh(zJ_+/2) \cosh(zN_+/2)\} - \{N_-, \sinh(zJ_+/2) \sinh(zN_+/2)\}$$
$$[J_3, N_+] = \frac{4}{z} \sinh(zN_+/2) \cosh(zJ_+/2) \tag{3.17}$$
$$[J_3, N_-] = -\{N_-, \cosh(zJ_+/2) \cosh(zN_+/2)\} - \{J_-, \sinh(zJ_+/2) \sinh(zN_+/2)\}$$
$$[N_3, N_\pm] = [J_3, J_\pm] \qquad [N_3, J_\pm] = [J_3, N_\pm]$$
$$[J_+, J_-] = [N_+, N_-] = J_3 \qquad [J_\pm, N_\mp] = \pm N_3 \qquad [J_m, N_m] = 0 \qquad m = \pm, 3$$

(where $\{X, Y\} = XY + YX$ denotes the anticommutator of X and Y) define the Hopf algebra $U_z^{(n)} so(2, 2) = U_z^{(n)} so(2, 1) \oplus U_{-z}^{(n)} so(2, 1)$.

The classical r-matrix associated to $U_z^{(n)} so(2, 2)$ is

$$r = J_3^1 \wedge J_+^1 - J_3^2 \wedge J_+^2 = \tfrac{1}{2}(J_3 \wedge N_+ + N_3 \wedge J_+). \tag{3.18}$$

This is a (skew) solution of the CYBE. The cocommutator (3.10) defined here by (3.18) is consistent with the coproduct (3.15). Note that the r-matrices (3.14) and (3.18) are non-degenerate and degenerate, respectively, and preserve the original character of their components as far as the Yang–Baxter equation is concerned.

4. Quantum contractions

The aim of this section is to use both the standard and the non-standard quantum deformations $U_z^{(m)} so(2,2)$ ($m = n, s$) we have just described in order to obtain quantum deformations of the graded contractions of $so(2,2)$ studied in section 2. We first extend the definitions of classical involutions (2.12) and contractions (2.24) to the quantum case by assuming that they act on the algebra generators as in the classical case, and that their behaviour on z is determined in each case in such a way that the exponents in $e^{-zN_3/2}$ (see equation (3.11)) or $e^{-zN_+/2}$ (see equation (3.15)) are invariant under these q-involutions and contractions.

Explicitly, this means that the classical expressions (2.12) and (2.24) should be augmented to (with w as the contracted deformation parameter)

$$S_{q(s)}^{(\epsilon_1,\epsilon_2)} : (J, P_1, P_2, C_1, C_2, D; z) \to (\epsilon_1 J, \epsilon_2 P_1, \epsilon_1\epsilon_2 P_2, \epsilon_2 C_1, \epsilon_1\epsilon_2 C_2, D; \epsilon_1 z) \qquad (4.1)$$

$$(J, P_1, P_2, C_1, C_2, D; w) = \Gamma_{q(s)}^{(\mu_1,\mu_2,\mu_3)}(N_3/2, J_+, N_+, -J_-, N_-, J_3/2; z)$$
$$:= (\sqrt{\mu_1\mu_3}N_3/2, \sqrt{\mu_2\mu_3}J_+, \sqrt{\mu_1\mu_2}N_+,$$
$$-\sqrt{\mu_2\mu_3}J_-, \sqrt{\mu_1\mu_2}N_-, J_3/2; z/\sqrt{\mu_1\mu_3}) \qquad (4.2)$$

for the standard deformation, and to

$$S_{q(n)}^{(\epsilon_1,\epsilon_2)} : (J, P_1, P_2, C_1, C_2, D; z) \to (\epsilon_1 J, \epsilon_2 P_1, \epsilon_1\epsilon_2 P_2, \epsilon_2 C_1, \epsilon_1\epsilon_2 C_2, D; \epsilon_1\epsilon_2 z) \qquad (4.3)$$

$$(J, P_1, P_2, C_1, C_2, D; w) = \Gamma_{q(n)}^{(\mu_1,\mu_2,\mu_3)}(N_3/2, J_+, N_+, -J_-, N_-, J_3/2; z)$$
$$:= (\sqrt{\mu_1\mu_3}N_3/2, \sqrt{\mu_2\mu_3}J_+, \sqrt{\mu_1\mu_2}N_+,$$
$$-\sqrt{\mu_2\mu_3}J_-, \sqrt{\mu_1\mu_2}N_-, J_3/2; z/\sqrt{\mu_1\mu_2}) \qquad (4.4)$$

for the non-standard one. We have in both cases a $\mathbb{Z}_2 \times \mathbb{Z}_2$ abelian group of q-involutions. The q-deformed Hopf algebras corresponding to the classical contractions (2.17) of $so(2,2)$ can be directly obtained by applying the transformations (4.2) (resp. (4.4)) to (3.11)–(3.13) (resp. to (3.15)–(3.17)). Due to the classical origin of these transformations, the q-deformed commutation relations are always well defined after (4.2) (resp. (4.4)) has been applied. This does not happen, neither for the coproduct nor for the r-matrix, in a different way for each case. Table 1 sum up the results about the existence and properties of these standard and non-standard contracted quantum algebras.

Some remarks are in order:

- $U_w^{(s)} g_{(\mu_1,+1,\mu_3)}$. We get a well defined Hopf structure, which comes from an r-matrix. This is displayed in table 1 by a $\sqrt{}$. This family is studied in section 4.1.
- $U_w^{(s)} g_{(\mu_1,0,\mu_3)}$. In this case the limit $\mu_2 \to 0$ of the transformation (4.2) applied to the standard deformation of $so(2,2)$ originates a Hopf algebra which has a deformed coproduct and classical (i.e. non-deformed) commutation rules. This algebra is a deformation of a bi-algebra which is not a coboundary: there is no r-matrix for it, as one could expect from the fact that wr (with r given by (3.14)) diverges when μ_2 goes to zero. In table 1 this is shown by a $(\sqrt{})$.
- $U_w^{(n)} g_{(\mu_1,\mu_2,+1)}$. We also get a well defined Hopf structure, which comes from an r-matrix. This family is studied in section 4.2.

Table 1. Existence of standard and non-standard deformations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded contracted algebras of $so(2, 2)$. (A $\checkmark$ marks the existence of $U_w g$.)

Lie algebra $g_{(\mu_1,\mu_2,\mu_3)}$	(μ_1, μ_2, μ_3)		$U_w^{(s)} g_{(\mu_1,\mu_2,\mu_3)}$	$U_w^{(n)} g_{(\mu_1,\mu_2,\mu_3)}$
$so(2, 2)$	$(+, +, +)$	$(-, -, -)$	$\checkmark$	$\checkmark$
$so(2, 2)$	$(+, -, +)$	$(-, +, -)$	$\checkmark$	$\checkmark$
$so(3, 1)$	$(+, +, -)$	$(-, -, +)$	$\checkmark$	$\checkmark$
$so(3, 1)$	$(-, +, +)$	$(+, -, -)$	$\checkmark$	$\checkmark$
$iso(2, 1)$	$(0, +, +)$	$(0, -, -)$	$\checkmark$	$\checkmark$
$iso(2, 1)$	$(+, +, 0)$	$(-, -, 0)$	$\checkmark$	
$iso(2, 1)$	$(0, -, +)$	$(0, +, -)$	$\checkmark$	$\checkmark$
$iso(2, 1)$	$(+, -, 0)$	$(-, +, 0)$	$\checkmark$	
$t_4(so(1, 1) \oplus so(1, 1))$	$(+, 0, +)$	$(-, 0, -)$	$(\checkmark)$	$\checkmark$
$t_4(so(2) \oplus so(1, 1))$	$(-, 0, +)$	$(+, 0, -)$	$(\checkmark)$	$\checkmark$
$iiso(1, 1)$	$(0, 0, +)$	$(0, 0, -)$	$(\checkmark)$	$\checkmark$
$iiso(1, 1)$	$(+, 0, 0)$	$(-, 0, 0)$	$(\checkmark)$	
$i'iso(1, 1)$	$(0, +, 0)$	$(0, -, 0)$	$\checkmark$	
$(\mathbb{R}^4 + \mathbb{R}) \oplus \mathbb{R}$	$(0, 0, 0)$		$(\checkmark)$	

- $U_w^{(n)} g_{(\mu_1,\mu_2,0)}$. In contradistinction with the standard case, the limit $\mu_3 \to 0$ of the transformation (4.4) applied to the non-standard deformation of $so(2, 2)$ *does not produce a Hopf algebra*, because the coproduct is not well defined in the limit $\mu_3 \to 0$.

As all graded contractions of $so(2, 2)$ are equivalent to one of type either (a/a0), or (b/b0), and on the other hand types (a0), (b0) are those with an undefined coproduct or r-matrix, we shall only deal with the quantum deformations arising from the standard family (a): $g_{(\mu_1,+1,\mu_3)}$, and from the non-standard family (b): $g_{(\mu_1,\mu_2,+1)}$.

4.1. The standard algebras $U_w^{(s)} g_{(\mu_1,+1,\mu_3)}$

Proposition 5 (The standard quantization of $g_{(\mu_1,+1,\mu_3)}$). When $\mu_2 = +1$ the transformation (4.2) gives rise to the quantum algebra $U_w^{(s)} g_{(\mu_1,+1,\mu_3)}$ with r-matrix $r = (P_1 \wedge C_2 - P_2 \wedge C_1)$ and defined by

$$\Delta J = 1 \otimes J + J \otimes 1 \qquad \Delta D = 1 \otimes D + D \otimes 1$$
$$\Delta P_1 = e^{-wJ} C_{-\mu_1\mu_3}(wD) \otimes P_1 + P_1 \otimes C_{-\mu_1\mu_3}(wD) e^{wJ}$$
$$\quad - e^{-wJ} S_{-\mu_1\mu_3}(wD) \otimes \mu_3 P_2 + \mu_3 P_2 \otimes S_{-\mu_1\mu_3}(wD) e^{wJ}$$
$$\Delta P_2 = e^{-wJ} C_{-\mu_1\mu_3}(wD) \otimes P_2 + P_2 \otimes C_{-\mu_1\mu_3}(wD) e^{wJ}$$
$$\quad - e^{-wJ} S_{-\mu_1\mu_3}(wD) \otimes \mu_1 P_1 + \mu_1 P_1 \otimes S_{-\mu_1\mu_3}(wD) e^{wJ} \qquad (4.5)$$
$$\Delta C_1 = e^{-wJ} C_{-\mu_1\mu_3}(wD) \otimes C_1 + C_1 \otimes C_{-\mu_1\mu_3}(wD) e^{wJ}$$
$$\quad + e^{-wJ} S_{-\mu_1\mu_3}(wD) \otimes \mu_3 C_2 - \mu_3 C_2 \otimes S_{-\mu_1\mu_3}(wD) e^{wJ}$$
$$\Delta C_2 = e^{-wJ} C_{-\mu_1\mu_3}(wD) \otimes C_2 + C_2 \otimes C_{-\mu_1\mu_3}(wD) e^{wJ}$$
$$\quad + e^{-wJ} S_{-\mu_1\mu_3}(wD) \otimes \mu_1 C_1 - \mu_1 C_1 \otimes S_{-\mu_1\mu_3}(wD) e^{wJ}$$
$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{2wJ} X e^{-2wJ} \qquad X \in \{J, P_i, C_i, D\} \qquad (4.6)$$
$$[J, P_1] = \mu_3 P_2 \qquad [J, P_2] = \mu_1 P_1 \qquad [P_1, P_2] = 0$$
$$[J, C_1] = \mu_3 C_2 \qquad [J, C_2] = \mu_1 C_1 \qquad [C_1, C_2] = 0$$

$$[P_1, C_1] = -\frac{1}{w}\mu_3\, S_{-\mu_1\mu_3}(2wD)\cosh(2wJ) \tag{4.7}$$

$$[P_2, C_2] = \frac{1}{w}\mu_1\, S_{-\mu_1\mu_3}(2wD)\cosh(2wJ)$$

$$[P_1, C_2] = [C_1, P_2] = \frac{1}{w}\sinh(2wJ)\, C_{-\mu_1\mu_3}(2wD)$$

$$[D, P_i] = P_i \qquad [D, C_i] = -C_i \qquad [D, J] = 0 \qquad i = 1, 2\,.$$

We recall that the generalized sine and cosine functions are given by [8]

$$C_{-\mu}(x) = \frac{e^{\sqrt{\mu}x} + e^{-\sqrt{\mu}x}}{2} \qquad S_{-\mu}(x) = \frac{e^{\sqrt{\mu}x} - e^{-\sqrt{\mu}x}}{2\sqrt{\mu}}\,. \tag{4.8}$$

Note that two non-trivial Hopf subalgebras with *undeformed* commutation brackets are contained in $U_w^{(s)}g_{(\mu_1,+1,\mu_3)}$: $\langle J, P_1, P_2, D\rangle$ and $\langle J, C_1, C_2, D\rangle$. However, neither $\langle J, P_1, P_2\rangle$ nor $\langle J, C_1, C_2\rangle$ are Hopf subalgebras; the generator D is required to close their coproducts. This fact is a general feature of the deformation procedure developed in [13] for higher-dimensional cases of the algebras $U_w^{(s)}g_{(\mu_1,+1,+1)}$, which can be interpreted as conformal algebras of flat two-dimensional spaces.

A long but straightforward computation leads to the next proposition.

Proposition 6. The centre of $U_w^{(s)}g_{(\mu_1,+1,\mu_3)}$ is generated by

$$C_1^q = \frac{C_{-\mu_1\mu_3}(2w)}{w^2}\left[\sinh^2(wJ)\,C_{-\mu_1\mu_3}^2(wD) + \mu_1\mu_3\,S_{-\mu_1\mu_3}^2(wD)\cosh^2(wJ)\right]$$

$$+ \frac{S_{-\mu_1\mu_3}(2w)}{4w}[-\mu_1(P_1C_1 + C_1P_1) + \mu_3(P_2C_2 + C_2P_2)] \tag{4.9}$$

$$C_2^q = \frac{C_{-\mu_1\mu_3}(2w)}{4w^2}\sinh(2wJ)\,S_{-\mu_1\mu_3}(2wD) + \frac{S_{-\mu_1\mu_3}(2w)}{4w}[P_1C_2 - C_1P_2]\,. \tag{4.10}$$

Finally we remark that the classical r-matrix is the same for all $U_w^{(s)}g_{(\mu_1,+1,\mu_3)}$ due to the invariance of the product wr under the transformation (4.2).

4.2. The non-standard algebras $U_w^{(n)}g_{(\mu_1,\mu_2,+1)}$

A similar approach applied to the non-standard q-deformation of $so(2, 2)$ leads to:

Proposition 7 (The non-standard quantization of $g_{(\mu_1,\mu_2,+1)}$). If we keep $\mu_3 = +1$, then we get a quantum algebra $U_w^{(n)}g_{(\mu_1,\mu_2,+1)}$ with r-matrix $r = J \wedge P_1 + D \wedge P_2$ given by

$$\Delta P_1 = 1 \otimes P_1 + P_1 \otimes 1 \qquad \Delta P_2 = 1 \otimes P_2 + P_2 \otimes 1$$

$$\Delta C_1 = e^{-wP_2/2}\,C_{-\mu_1}(wP_1/2) \otimes C_1 + C_1 \otimes C_{-\mu_1}(wP_1/2)\,e^{wP_2/2}$$
$$+ e^{-wP_2/2}\,S_{-\mu_1}(wP_1/2) \otimes C_2 - C_2 \otimes S_{-\mu_1}(wP_1/2)\,e^{wP_2/2}$$

$$\Delta C_2 = e^{-wP_2/2}\,C_{-\mu_1}(wP_1/2) \otimes C_2 + C_2 \otimes C_{-\mu_1}(wP_1/2)\,e^{wP_2/2}$$
$$+ e^{-wP_2/2}\,S_{-\mu_1}(wP_1/2) \otimes \mu_1 C_1 - \mu_1 C_1 \otimes S_{-\mu_1}(wP_1/2)\,e^{wP_2/2} \tag{4.11}$$

$$\Delta J = e^{-wP_2/2}\,C_{-\mu_1}(wP_1/2) \otimes J + J \otimes C_{-\mu_1}(wP_1/2)\,e^{wP_2/2}$$
$$- e^{-wP_2/2}\,S_{-\mu_1}(wP_1/2) \otimes \mu_1 D + \mu_1 D \otimes S_{-\mu_1}(wP_1/2)\,e^{wP_2/2}$$

$$\Delta D = e^{-wP_2/2}\,C_{-\mu_1}(wP_1/2) \otimes D + D \otimes C_{-\mu_1}(wP_1/2)\,e^{wP_2/2}$$
$$- e^{-wP_2/2}\,S_{-\mu_1}(wP_1/2) \otimes J + J \otimes S_{-\mu_1}(wP_1/2)\,e^{wP_2/2}$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{wP_2}\,X\,e^{-wP_2} \qquad X \in \{J, P_i, C_i, D\} \tag{4.12}$$

$$[J, P_1] = \frac{2}{w} \sinh(wP_2/2)\, C_{-\mu_1}(wP_1/2)$$

$$[J, P_2] = \frac{2}{w}\mu_1\, S_{-\mu_1}(wP_1/2)\cosh(wP_2/2)$$

$$[J, C_1] = \tfrac{1}{2}\{C_2,\ C_{-\mu_1}(wP_1/2)\cosh(wP_2/2)\} - \tfrac{1}{2}\mu_1\{C_1,\ S_{-\mu_1}(wP_1/2)\sinh(wP_2/2)\}$$

$$[J, C_2] = \tfrac{1}{2}\mu_1\{C_1,\ C_{-\mu_1}(wP_1/2)\cosh(wP_2/2)\} - \tfrac{1}{2}\mu_1\{C_2,\ S_{-\mu_1}(wP_1/2)\sinh(wP_2/2)\}$$

$$[P_1, P_2] = [C_1, C_2] = 0 \qquad [P_1, C_2] = [C_1, P_2] = 2\mu_2 J \qquad (4.13)$$

$$[P_1, C_1] = -2\mu_2 D \qquad [P_2, C_2] = 2\mu_1\mu_2 D \qquad [D, J] = 0$$

$$[D, P_1] = \frac{2}{w} S_{-\mu_1}(wP_1/2)\cosh(wP_2/2)$$

$$[D, P_2] = \frac{2}{w}\sinh(wP_2/2)\, C_{-\mu_1}(wP_1/2)$$

$$[D, C_1] = -\tfrac{1}{2}\{C_1,\ C_{-\mu_1}(wP_1/2)\cosh(wP_2/2)\} + \tfrac{1}{2}\{C_2,\ S_{-\mu_1}(wP_1/2)\sinh(wP_2/2)$$

$$[D, C_2] = -\tfrac{1}{2}\{C_2,\ C_{-\mu_1}(wP_1/2)\cosh(wP_2/2)\} + \tfrac{1}{2}\mu_1\{C_1,\ S_{-\mu_1}(wP_1/2)\sinh(wP_2/2)\}\,.$$

The generators $\langle J, P_1, P_2, D\rangle$ close a Hopf subalgebra with deformed commutation relations. The invariance of the product wr under (4.4) for $\mu_3 = +1$ explains again the fact that the classical r-matrix is the same for all $U_w^{(n)} g_{(\mu_1,\mu_2,+1)}$. Central elements can be stated as follows:

Proposition 8. The centre of $U_w^{(n)} g_{(\mu_1,\mu_2,+1)}$ is generated by

$$C_1^q = \mu_2 J^2 + \mu_1\mu_2 D^2 - \frac{1}{w}\mu_1\{C_1,\ S_{-\mu_1}(wP_1/2)\cosh(wP_2/2)\}$$

$$+\frac{1}{w}\{C_2,\ \sinh(wP_2/2)\, C_{-\mu_1}(wP_1/2)\} + \mu_1\mu_2\, C_{-\mu_1}(wP_1)\cosh(wP_2) \quad (4.14)$$

$$C_2^q = \mu_2 J D + \frac{1}{w} S_{-\mu_1}(wP_1/2)\cosh(wP_2/2)C_2$$

$$-\frac{1}{w}C_1\sinh(wP_2/2)\, C_{-\mu_1}(wP_1/2) + \tfrac{1}{2}\mu_1\mu_2\, S_{-\mu_1}(wP_1)\sinh(wP_2)\,. \quad (4.15)$$

5. Non-standard quantum kinematical and conformal algebras

The non-standard structure $U_w^{(n)} g_{(\mu_1,\mu_2,+1)}$ gives rise to a new set of physically interesting Hopf algebras. Their new properties can be highlighted by comparing them, when possible, to the standard ones. This is the case for the three upper front edge algebras $g_{(\mu_1,+1,+1)}$ of figure 1, with $\mu_1 = 1, 0, -1$, that are isomorphic to $so(2, 2)$, $iso(2, 1)$ and $so(3, 1)$, and support both standard and non-standard quantum deformations.

These three algebras have two different realizations at the classical level. The first one is as isometry algebras of motion groups of $(2 + 1)$ Lorentzian spaces with constant curvature (anti-de Sitter, Minkowski and de Sitter spaces). The second realization arises if we consider them as conformal algebras of flat spaces: the $(1 + 1)$ Minkowski and Galilean spaces and the 2D Euclidean plane.

5.1. A 'null plane' deformation of (2+1) Poincaré and de Sitter algebras

The structure of the standard deformation of algebras $g_{(\mu_1,1,1)}$ is more clearly appreciated in a new physical basis, $\{H, T_1, T_2, K_1, K_2, L\}$ generating the time translation, space translations,

boosts and space rotation, respectively. The required change of basis from the former one $\{J, P_1, P_2, C_1, C_2, D\}$ is

$$H = \tfrac{1}{2}(P_2 - C_2) \qquad T_1 = \tfrac{1}{2}(P_2 + C_2) \qquad T_2 = J$$
$$K_1 = D \qquad K_2 = \tfrac{1}{2}(C_1 - P_1) \qquad L = \tfrac{1}{2}(C_1 + P_1). \tag{5.1}$$

Here we do not give the standard coproduct or the deformed commutators (which can easily be obtained from (4.5)–(4.8)) but simply remark that the standard deformation $U_w^{(s)} g_{(\mu_1,1,1)}$ has T_2 and K_1 as primitive generators and was studied in [8].

Things are dramatically different for the non-standard deformation since the primitive generators are P_1 and P_2, therefore in the basis (5.1) we would find a 'mixture' of primitive and non-primitive generators. Hence, the most adapted basis to write down the Hopf algebra is $\{J, P_i, C_i, D\}$. When these generators are expressed in terms of $\{H, T_i, K_i, L\}$ we get the null plane basis $\{T_+, T_-, T_2, K_1, E, F\}$, where the new generators T_+, T_-, E, F are given in terms of the physical basis by the known expressions

$$T_+ = T_1 + H \equiv P_2 \qquad T_- = T_1 - H \equiv C_2$$
$$E = L - K_2 \equiv P_1 \qquad F = L + K_2 \equiv C_1. \tag{5.2}$$

By means of (5.2) and after the specialization $\mu_1 = 0$ and $\mu_2 = \mu_3 = 1$ in the formulae (4.11)–(4.15) we obtain the coproduct, counit, antipode, deformed Lie brackets and Casimirs defining the $(2+1)$ non-standard q-Poincaré algebra:

$$\Delta T_+ = 1 \otimes T_+ + T_+ \otimes 1 \qquad \Delta E = 1 \otimes E + E \otimes 1$$
$$\Delta T_- = e^{-wT_+/2} \otimes T_- + T_- \otimes e^{wT_+/2} \qquad \Delta T_2 = e^{-wT_+/2} \otimes T_2 + T_2 \otimes e^{wT_+/2}$$
$$\Delta K_1 = e^{-wT_+/2} \otimes K_1 + K_1 \otimes e^{wT_+/2} - \tfrac{1}{2}w \left(e^{-wT_+/2} E \otimes T_2 + T_2 \otimes E\, e^{wT_+/2} \right) \tag{5.3}$$
$$\Delta F = e^{-wT_+/2} \otimes F + F \otimes e^{wT_+/2} + \tfrac{1}{2}w \left(e^{-wT_+/2} E \otimes T_- - T_- \otimes E\, e^{wT_+/2} \right)$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{wT_+} X\, e^{-wT_+} \qquad X \in \{T_+, T_-, T_2, K_1, E, F\} \tag{5.4}$$

$$[T_2, E] = \frac{2}{w}\sinh(wT_+/2) \qquad [T_2, F] = T_- \cosh(wT_+/2)$$

$$[E, T_-] = [F, T_+] = 2T_2 \qquad [E, F] = -2K_1$$

$$[K_1, T_+] = \frac{2}{w}\sinh(wT_+/2) \qquad [K_1, T_-] = -T_- \cosh(wT_+/2) \tag{5.5}$$

$$[K_1, E] = E \cosh(wT_+/2) \qquad [K_1, F] = -F \cosh(wT_+/2) + \tfrac{1}{2}wE\, T_- \sinh(wT_+/2)$$

$$[T_2, T_\pm] = [T_+, T_-] = [T_2, K_1] = [E, T_+] = [F, T_-] = 0$$

$$C_1^q = T_2^2 + \frac{2}{w} T_- \sinh(wT_+/2) \tag{5.6}$$

$$C_2^q = T_2 K_1 + \tfrac{1}{2}E\, T_- \cosh(wT_+/2) - \frac{1}{w} F \sinh(wT_+/2). \tag{5.7}$$

Since the coproduct of K_1 involves E and T_2, the generators K_1, T_+ and T_- do not span a Hopf subalgebra; despite this fact, their commutation relations close a deformed $(1+1)$ Poincaré algebra [14]. On the contrary, the subset $\langle K_1, T_+, T_2, E \rangle$, whose elements generate classically the isotopy subalgebra of the null plane $x_- = 0$ [15], does span a Hopf subalgebra which can be interpreted as a $(1+1)$ q-Galilei algebra $\langle T_+, T_2, E \rangle$ enlarged with a dilation $\langle K_1 \rangle$.

A similar structure can readily be obtained when $\mu_1 = \pm 1$, leading to the quantum non-standard de Sitter algebras. In these cases, the curvature of the spacetime equals to $-\mu_1$.

The fact that the kinematical part of the null plane description (the isotopy subalgebra of the null plane) is preserved as a Hopf subalgebra under this non-standard deformation is rather remarkable. This fact could be interesting as a guide for the physical interpretation of this 'null plane deformation' of (2+1) Poincaré algebra, which should be related to situations where the classical null plane dynamics are relevant (for instance, when the ultrarelativistic limit is involved [16]).

5.2. Conformal algebras in two dimensions

The Lie algebra of $(1 + 1)$ affine groups, $\langle J, P_1, P_2 \rangle$, with commutators

$$[J, P_1] = P_2 \qquad [J, P_2] = \mu_1 P_1 \qquad [P_1, P_2] = 0 \tag{5.8}$$

reproduces the Euclidean, Galilei and Poincaré algebras for $\mu_1 < 0, = 0, > 0$, respectively. All the three algebras can be extended by adding a dilation generator, D, and two special conformal generators, C_1 and C_2, which close the algebra $g_{(\mu_1,1,1)}$. We therefore get the second realization above referred to for these three algebras (the ones allowing both types of deformation) as conformal algebras in two dimensions.

The standard deformation has J and D as primitive generators, and is given by (4.5)–(4.8). We do not elaborate upon the comments made in section 4.1.

However, the non-standard deformation is rather different, since now *both* translations P_1 and P_2 are *primitive*. While in the 'standard' case both pairs $\{P_1, P_2\}$ and $\{C_1, C_2\}$ enter on a completely symmetrical footing, this is no longer the case now, because C_1 and C_2 are not primitive. The generators $\langle J, P_1, P_2, D \rangle$ span a Hopf subalgebra with *deformed commutation rules*:

$$\Delta P_1 = 1 \otimes P_1 + P_1 \otimes 1 \qquad \Delta P_2 = 1 \otimes P_2 + P_2 \otimes 1$$
$$\Delta J = e^{-wP_2/2} C_{-\mu_1}(wP_1/2) \otimes J + J \otimes C_{-\mu_1}(wP_1/2) e^{wP_2/2}$$
$$\qquad - e^{-wP_2/2} S_{-\mu_1}(wP_1/2) \otimes \mu_1 D + \mu_1 D \otimes S_{-\mu_1}(wP_1/2) e^{wP_2/2}$$
$$\Delta D = e^{-wP_2/2} C_{-\mu_1}(wP_1/2) \otimes D + D \otimes C_{-\mu_1}(wP_1/2) e^{wP_2/2}$$
$$\qquad - e^{-wP_2/2} S_{-\mu_1}(wP_1/2) \otimes J + J \otimes S_{-\mu_1}(wP_1/2) e^{wP_2/2} \tag{5.9}$$
$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{wP_2} X e^{-wP_2} \qquad X \in \{J, P_i, D\}$$
$$[J, P_1] = \frac{2}{w} \sinh(wP_2/2) C_{-\mu_1}(wP_1/2) \qquad [J, P_2] = \frac{2}{w} \mu_1 S_{-\mu_1}(wP_1/2) \cosh(wP_2/2)$$
$$[D, P_1] = \frac{2}{w} S_{-\mu_1}(wP_1/2) \cosh(wP_2/2) \qquad [D, P_2] = \frac{2}{w} \sinh(wP_2/2) C_{-\mu_1}(wP_1/2)$$
$$[P_1, P_2] = 0 \qquad [D, J] = 0.$$

Note that $\langle J, C_1, C_2, D \rangle$ is not a Hopf subalgebra. These properties prompt to focus attention to the classical differential realization of conformal algebras given by

$$P_1 = \partial_1 \qquad P_2 = \partial_2$$
$$J = -\mu_1 x_2 \partial_1 - x_1 \partial_2 + B$$
$$D = -x_1 \partial_1 - x_2 \partial_2 + A + 1 \tag{5.10}$$
$$C_1 = (x_1^2 + \mu_1 x_2^2) \partial_1 + 2x_1 x_2 \partial_2 - 2(A + 1)x_1 - 2Bx_2$$
$$C_2 = -(x_1^2 + \mu_1 x_2^2) \partial_2 - 2\mu_1 x_1 x_2 \partial_1 + 2\mu_1(A + 1)x_2 + 2Bx_1$$

where A and B are arbitrary real constants (for $A = -1$ and $B = 0$ this reproduces the fundamental fields for the local action of the conformal group on the 2D space). The non-standard q-deformed version of this realization gives us

$$P_1 = \partial_1 \qquad P_2 = \partial_2$$

$$\begin{aligned}
J = &-\mu_1 x_2 \frac{2}{w} \, \mathrm{S}_{-\mu_1}(w\partial_1/2)\cosh(w\partial_2/2) - x_1 \frac{2}{w}\,\mathrm{C}_{-\mu_1}(w\partial_1/2)\sinh(w\partial_2/2) \\
&+ B + \mu_1 \, \mathrm{S}_{-\mu_1}(w\partial_1/2)\sinh(w\partial_2/2)
\end{aligned}$$

$$\begin{aligned}
D = &-x_1 \frac{2}{w}\,\mathrm{S}_{-\mu_1}(w\partial_1/2)\cosh(w\partial_2/2) - x_2 \frac{2}{w}\,\mathrm{C}_{-\mu_1}(w\partial_1/2)\sinh(w\partial_2/2) \\
&+ A + \mathrm{C}_{-\mu_1}(w\partial_1/2)\cosh(w\partial_2/2)
\end{aligned} \qquad (5.11)$$

$$\begin{aligned}
C_1 = &(x_1^2 + \mu_1 x_2^2)\frac{2}{w}\,\mathrm{S}_{-\mu_1}(w\partial_1/2)\cosh(w\partial_2/2) + x_1 x_2 \frac{4}{w}\,\mathrm{C}_{-\mu_1}(w\partial_1/2)\sinh(w\partial_2/2) \\
&- 2x_1[A + \mathrm{C}_{-\mu_1}(w\partial_1/2)\cosh(w\partial_2/2)] \\
&- 2x_2[B + \mu_1 \, \mathrm{S}_{-\mu_1}(w\partial_1/2)\sinh(w\partial_2/2)]
\end{aligned}$$

$$\begin{aligned}
C_2 = &-(x_1^2 + \mu_1 x_2^2)\frac{2}{w}\,\mathrm{C}_{-\mu_1}(w\partial_1/2)\sinh(w\partial_2/2) - \mu_1 x_1 x_2 \frac{4}{w}\,\mathrm{S}_{-\mu_1}(w\partial_1/2)\cosh(w\partial_2/2) \\
&+ 2\mu_1 x_2[A + \mathrm{C}_{-\mu_1}(w\partial_1/2)\cosh(w\partial_2/2)] \\
&+ 2x_1[B + \mu_1 \, \mathrm{S}_{-\mu_1}(w\partial_1/2)\sinh(w\partial_2/2)] \, .
\end{aligned}$$

Note also that the symmetric q-derivative

$$\begin{aligned}
D_q f(x) := \frac{\sinh(w\partial_x/2)}{w/2} f(x) &= \frac{\exp(w\partial_x/2) - \exp(-w\partial_x/2)}{w} f(x) \\
&= \frac{f(x + w/2) - f(x - w/2)}{w}
\end{aligned} \qquad (5.12)$$

is naturally contained in the realization (5.11) for both P_1 and P_2 generators.

We recall that within the known standard deformations of $(1 + 1)$ and $(2 + 1)$ algebras, such a discretization appears only in *one* spatial direction [4, 5, 8]. Therefore a 'conformal' approach to this problem seems to be promising as it would allow a kind of complete discretization of the spacetime in $(1 + 1)$ dimensions.

6. Concluding remarks

As a general result, we emphasize that a systematic use of the theory of graded contractions provides a well defined and encompassing framework to study q-deformations of some real non-semisimple algebras in a straightforward way. Both standard and non-standard deformations of $so(2, 2)$ introduced in this paper generate by contraction quantum deformations of some relevant kinematical and conformal groups. Some of them are new and others coincide with already known quantum algebras as the q-Poincaré and q-de Sitter algebras obtained in [8].

A point worth stressing is that, in general, the contraction process needs a careful examination; for each $so(2, 2)$ deformation not all possible classical graded contractions induce quantum deformations for the contracted algebras. The same analysis can be performed for the quantum R-matrices. This fact is related to some problems which arise when contractions of quantum groups are made in a naive way.

By consistently using only real forms some contraction processes found in the literature can be also clarified. Of course, if complex coefficients are allowed in the changes of basis, such as was done in [4, 5], then further possibilities are opened. In this way and starting from any standard quantum $iso(2, 1)$ algebra we could get the $(2 + 1)$ κ-Poincaré [17, 18], which does not appear, as such, in our scheme. Another example along this line is provided by the contraction sequence $SO(4)_q \rightarrow E(3)_q \rightarrow G(2)_q$ studied in

[4,5], that corresponds, in our context of real forms, to the standard quantum algebras $so(2,2)_q \to iso(2,1)_q \to i'iso(1,1)_q$, this is, $U_w^{(s)}g_{(1,1,1)} \to U_w^{(s)}g_{(0,1,1)} \to U_w^{(s)}g_{(0,1,0)}$. In this sense, it is important to recall that as a real form, the algebra $i'iso(1,1)_q$ is not isomorphic to the $(2+1)$ Galilean algebra: the latter does not include a central generator while the algebra $i'iso(1,1)$ does.

A way opened by this paper would consist in the construction of the quantum groups corresponding to the non-standard family of algebras. For all of them, the existence of a star product that quantizes their classical Poisson–Lie structures is guaranteed (their r-matrices exist and are degenerate). Moreover, a solution of the quantum YBE linked to each non-standard quantum algebra can be obtained as a bi-product of the bi-differential operator that defines the star product on the group [19]. Also, the natural link of the non-standard deformation of the Poincaré algebra to the null plane basis and its possible physical interpretation should require further study. Should this deformation survive for the $(3 + 1)$ case, we would get a new quantum $(3 + 1)$ Poincaré algebra whose features would be certainly different from the known κ-Poincaré algebra [17, 18]. Work on these lines is currently in progress.

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